

Computation of Greeks Using Malliavin Calculus

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Abstract

The objective of this paper is to explore application of Malliavin calculus techniques to the problem of estimating greeks of financial derivative contracts. In the core of the method is the integration by parts formula of Malliavin calculus. After a short introduction to the theory of Malliavin calculus, the technique is described on different types of option contracts, such as vanilla calls, digital, Asian and some barrier and lookback options.

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Chapter 1

Introduction

In this work we study application of Malliavin Calculus to estimation of financial quantities called greeks. The results obtained using this technique allow further efficient use of Monte-Carlo methods to numerically estimate, in particular, sensitivity of prices of derivative option contracts to changes in parameters of a model under consideration. More specifically, option contracts are described by their payoff functions, Φ , and prices of such contracts are equal to expectations of the form $E[e^{-rT}\Phi]$, where constants r and T are the interest rate and the time of expiry of the contract, respectively. It is enough to say here that the function Φ , in general, depends on the price of the underlying, which, in turn, depends on parameters of the model. Then greeks are just derivatives of the mentioned expectation with respect to these parameters.

For example, when the underlying of an option is a share of stock, a derivative of the price of such option with respect to the price of the stock at the moment of initiation of the contract is called Delta, $\Delta = \partial_{S_0}E[e^{-rT}\Phi]$, and a derivative of the price with respect to the volatility of the underlying stock is called Vega, $v = \partial_{\sigma}E[e^{-rT}\Phi]$. A strong motivation to consider such quantities comes, among other areas of financial research, from risk management. Today it is vital for financial risk managers and traders to understand sensitivity of values of portfolios of options to small changes in underlying parameters, so that different components of risk can be treated separately and portfolios can be protected or hedged from potential exposures. A very good and extensive introduction to

derivative contracts and greeks can be found in [4].

In practice several methods are currently used to obtain such estimates. In the case when the density of the random variable in the expectation is known, such estimates can be computed using the likelihood method, described in [1]. As in many practical situations values of derivative contracts can be described by partial differential equations, the usual numerical methods for PDEs, such as finite difference and finite element methods, find various applications. But complications occur when the density in the expectation is not known or the payoff function is not smooth enough for these methods to produce accurate results, which actually is the case for a large group of so called exotic options. This is the case where Integration by Parts formula (IBP) of Malliavin calculus proved to be very effective, both in providing fairly simple computations and good convergence rates of the subsequent Monte Carlo simulations. A good description of all these methods and discussion of rates of convergence can be found in [5] and [2]. In what follows we will concentrate on the method using Malliavin calculus.

The following work is organized as follows. First, in chapter 2 basic concepts and results of Malliavin calculus needed for the method are introduced. Then, in chapter 3 various computations related to greeks of some simple, or so called vanilla, options and exotic Asian options are carried out. Lastly, in chapter 4 one more sophisticated method of estimation of greeks of a particular group of exotic options is analyzed. Throughout the paper knowledge of basic Ito calculus is assumed.

Chapter 2

Introduction to Malliavin Calculus for Finance

In this section we present important definitions and some basic results of Malliavin calculus needed in financial applications. The exposition here mostly follows [6] and [8] (proofs of all of the following results can be found there). Another similar, though shorter, introductory articles are [7] and [9].

2.1 The Wiener-Ito Chaos Expansion

Malliavin calculus defines derivatives of functions on Wiener space and can be thought of as a theory of integration by parts on this space. We start by developing the fundamental Wiener-Ito chaos expansion theorem. To be able to state the result we introduce some notation. Let $W_t = W_t(\omega)$, $t \geq 0$, $\omega \in \Omega$, be a 1-dimensional standard Brownian motion defined on a probability space (Ω, \mathcal{F}, P) . For $t \geq 0$, let \mathcal{F}_t be a σ -algebra generated by $W_s(\cdot)$, $0 \leq s \leq t$, and let T be a positive constant.

We say that a function $g : [0, T]^n \rightarrow \mathbb{R}$ is from a space of symmetric square integrable functions, $\widehat{L}^2([0, T]^n)$, if g is symmetric, i.e., $g(x_{\sigma_1}, \dots, x_{\sigma_n}) = g(x_1, \dots, x_n)$ for all permutations σ of

$(1, \dots, n)$, and

$$\|g\|_{L^2([0,T]^n)}^2 = \int_{[0,T]^n} g^2(x_1, \dots, x_n) dx_1 \dots dx_n < \infty.$$

Properties of such functions then allow us to consider, for $g \in \widehat{L}^2([0,T]^n)$, the n -fold iterated Ito integral

$$J_n(g) = \int_0^T \int_0^{t_n} \dots \int_0^{t_3} \left(\int_0^{t_2} g(t_1, \dots, t_n) dW_{t_1} \right) dW_{t_2} \dots dW_{t_{n-1}} dW_{t_n},$$

where $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T$, $n \geq 1$ (to avoid the diagonal), and $J_0(g) = g$ if g is constant. We can extend such integrals to define

$$I_n(g) = \int_{[0,T]^n} g(t_1, \dots, t_n) dW_t^{\otimes n} = n! J_n(g).$$

Now we are ready for

Theorem 2.1.1. *Let ϕ be an \mathcal{F}_T -measurable square integrable random variable, i.e.,*

$$\|\phi\|_{L^2(\Omega)}^2 = E_P[\phi^2] < \infty.$$

Then there exists a unique sequence $\{f_n\}_{n=0}^\infty$ of deterministic functions $f_n \in \widehat{L}^2([0,T]^n)$, such that

$$\phi(\omega) = \sum_{n=0}^{\infty} I_n(f_n),$$

with convergence considered in $L^2(P)$.

Examples 1. (all of the following examples are taken from exercises from [8])

(a) If $F(\omega) = W_{t_0}$, $t_0 \in [0, T]$ fixed, then the chaos expansion of F is

$$F(\omega) = \int_0^T \mathbb{1}_{[0, t_0]}(s) dW_s, \text{ i.e., } f_0 = 0, f_1 = \mathbb{1}_{[0, t_0]}, f_n = 0 \text{ for } n \geq 2.$$

(b) If $F(\omega) = \int_0^T g(s) dW_s$, where a deterministic function $g \in L^2[0, T]$, then F is already in the form of chaos expansion with $f_0 = 0$, $f_1 = g$, $f_n = 0$ for $n \geq 2$.

(c) Suppose $F = W_{t_0}^2$, $t_0 \in [0, T]$ fixed. Then, since, using Ito formula,

$$W_{t_0}^2 = 2 \int_0^{t_0} W_s dW_s + t_0$$

and

$$\int_0^{t_0} W_{t_2} dW_{t_2} = \int_0^{t_0} \int_0^{t_2} dW_{t_1} dW_{t_2} = \int_0^T \int_0^{t_2} \mathbb{1}_{[0, t_0]}(t_1) \mathbb{1}_{[0, t_0]}(t_2) dW_{t_1} dW_{t_2},$$

the chaos expansion of F is

$$F(\omega) = t_0 + 2 \int_0^T \int_0^{t_2} \mathbb{1}_{[0, t_0]}(t_1) \mathbb{1}_{[0, t_0]}(t_2) dW_{t_1} dW_{t_2},$$

$$i.e., \quad f_0 = t_0, \quad f_1 = 0, \quad f_2(t_1, t_2) = \mathbb{1}_{[0, t_0]}(t_1) \mathbb{1}_{[0, t_0]}(t_2), \quad f_n = 0 \text{ for } n \geq 3.$$

2.2 The Skorohod Integral

In what follows the notion of a Skorohod integral is introduced. We consider $u_t(\omega)$, $\omega \in \Omega$, $t \in [0, T]$, a (t, ω) -measurable stochastic process, such that, for all $t \in [0, T]$,

$$u_t \text{ is } \mathcal{F}_T\text{-adapted and } E[u_t^2(\omega)] < \infty.$$

We can then apply the Wiener-Ito chaos expansion theorem to obtain, for all $t \in [0, T]$,

$$u_t(\omega) = \sum_{n=0}^{\infty} I_n(f_{n, t}(\cdot)),$$

where $f_{n, t}(t_1, \dots, t_n) \in \widehat{L}^2([0, T]^n)$. Since in this case the functions $f_{n, t}(\cdot)$ depend on the parameter t , we denote $f_{n, t}(\cdot) = f_n(\cdot, t)$.

Definition 1. If $u_t(\omega)$ is a stochastic process satisfying conditions above with the chaos expansion

$$u_t(\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)),$$

then the Skorohod integral of u_t is defined to be

$$\delta(u) = \int_0^T u_t \delta W_t = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n),$$

where \tilde{f}_n is a symmetrization of $f_n(t_1, \dots, t_n, t)$ as a function of $n+1$ variables, and when such a series is convergent in $L^2(P)$. Another equivalent definition of this operator will be presented later in this section after Malliavin derivatives are introduced.

It is known that a process u_t is Skorohod-integrable, in which case we write $u \in \text{Dom}(\delta)$, if and only if

$$E[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L^2([0,T]^{n+1})}^2 < \infty.$$

To relate this new concept to the theory of Ito integration, we mention

Theorem 2.2.1. *If we assume that a stochastic process $u_t(\omega)$ is such that*

$$u_t \text{ is } \mathcal{F}_t\text{-adapted for all } t \in [0, T]$$

and

$$E \left[\int_0^T u_t^2(\omega) dt \right] < \infty,$$

then $u \in \text{Dom}(\delta)$ and

$$\delta(u) = \int_0^T u_t dW_t.$$

Examples 2.

(a) Let $u_t = W_t$, which is an \mathcal{F}_t -adapted process by definition. Then the application of Ito formula from example 1(c) and Theorem 2 yield

$$\delta(u) = \int_0^T W_s \delta W_s = \int_0^T W_s dW_s = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

(b) Let $u_t = \int_0^T g(s)dW_s$, where $g \in L^2[0, T]$ is a deterministic function. Then, following the example 1(b), for $f_1(t_1) = g(t_1)$,

$$\tilde{f}_1(t_1, t) = \frac{1}{2}(g(t_1) + g(t)),$$

so that

$$\begin{aligned} \delta(u) &= \int_0^T I_1(f_1(t_1, t)) \delta W_t = I_2(\tilde{f}_1(t_1, t)) = \\ &= \int_0^T \int_0^{t_2} g(t_1) dW_{t_1} dW_{t_2} + \int_0^T \int_0^{t_2} g(t_2) dW_{t_1} dW_{t_2}. \end{aligned}$$

To compute the first integral, we apply Ito formula to the product

$$W_T \left(\int_0^T g(t) dW_t \right) = \int_0^T \left(\int_0^s g(t) dW_t \right) dW_s + \int_0^T g(s) W_s dW_s + \int_0^T g(s) ds,$$

from where, finally,

$$\begin{aligned} \delta(u) &= W_T \left(\int_0^T g(s) dW_s \right) - \int_0^T g(s) W_s dW_s - \int_0^T g(s) ds + \\ &+ \int_0^T g(s) W_s dW_s = W_T \left(\int_0^T g(s) dW_s \right) - \int_0^T g(s) ds. \end{aligned}$$

In particular, this example implies that if $u_t = W_{t_1} = \int_0^T \mathbb{1}_{[0, t_1]}(s)dW_s$ for a fixed $t_1 \in [0, T]$, then

$$\delta(u) = W_T W_{t_1} - t_1.$$

2.3 The Derivative

We now proceed with the most important definition of this section. To set ground, we consider a Banach space $\Omega = C_0([0, T])$, a space of continuous functions ω with $\omega(0) = 0$. In this case, we can associate paths of a Brownian motion $W_t(\omega)$ with some element $\omega \in C_0([0, T])$, so that

$$W_t(\omega) = \omega(t).$$

In such a setting, the probability P associated with the Brownian motion becomes a measure μ , defined on cylinder sets of $C_0([0, T])$ by

$$\begin{aligned} \mu(\{\omega : \omega(t_1) \in F_1, \dots, \omega(t_m) \in F_m\}) &= P(W_{t_1} \in F_1, \dots, W_{t_m} \in F_m) = \\ &= \int_{F_1 \times \dots \times F_m} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \dots p(t_m - t_{m-1}, x_{m-1}, x_m) dx_1 \dots dx_m, \end{aligned}$$

where F_i are Borel subsets of \mathbb{R} , $0 \leq t_1 < t_2 < \dots < t_m$ and

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right) \text{ for } t > 0, x, y \in \mathbb{R}.$$

Such μ and Ω are called a Wiener measure and a Wiener space, respectively.

To define derivatives of random variables we also need the following two sets. First, consider Cameron-Martin space, the subset $H \subset \Omega$ of functions γ which can be written in the form

$$\gamma(t) = \int_0^t g(s) ds \text{ for some } g \in L^2([0, T]).$$

Further, let \mathbb{P} be a collection of random variables $F : \Omega \rightarrow \mathbb{R}$ of the form

$$F(\omega) = \phi(\theta_1, \dots, \theta_n),$$

where ϕ is a polynomial in n variables x_1, \dots, x_n and

$$\theta_i = \int_0^T f_i(t) dW_t \text{ for some deterministic function } f_i \in L^2([0, T]).$$

Elements of \mathbb{P} are called Wiener polynomials.

Definition 2. Let $F : \Omega \rightarrow \mathbb{R}$ be a random variable such that it has directional derivatives (in strong sense) in all directions $\gamma \in H$, i.e.,

$$D_\gamma F(\omega) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F(\omega + \varepsilon \gamma) - F(\omega))$$

exists in $L^2(\Omega)$ for all $\gamma \in H$. In addition, we assume that there is a $\psi_t(\omega) \in L^2([0, T] \times \Omega)$ such that

$$D_\gamma F(\omega) = \int_0^T \psi_t(\omega) g(t) dt.$$

Then the random variable F is said to be differentiable and the derivative of F is

$$D_t F(\omega) = \psi_t(\omega).$$

We denote the set of all such differentiable random variables by $\mathcal{D}_{1,2}$ with a norm

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|D_t F\|_{L^2([0,T] \times \Omega)}.$$

To ensure that a collection of random variables under consideration is closed under the $\|\cdot\|_{1,2}$ norm, we introduce yet another set $\mathbb{D}_{1,2}$, which is defined as a closure of the collection \mathbb{P} with respect to this norm. In other words, $F \in \mathbb{D}_{1,2}$ if there is a sequence $\{F_n\}$ of elements in \mathbb{P} such that

$$F_n \rightarrow F \text{ in } L^2(\Omega), n \rightarrow \infty,$$

and

$$\{D_t F_n\}_{n=0}^{\infty} \text{ converges in } L^2([0, T] \times \Omega).$$

We, thus, are ready for

Definition 3. If $F \in \mathbb{D}_{1,2}$, then the Malliavin derivative (or weak derivative) of F is defined to be

$$\mathcal{D}_t F = \lim_{n \rightarrow \infty} D_t F_n$$

and

$$\mathcal{D}_\gamma F = \langle \mathcal{D}F, g \rangle_{L^2([0,T])} = \int_0^T \mathcal{D}_t F \cdot g(t) dt \text{ for all } \gamma \in H.$$

Now several comments are in order. First, to clarify the relation between the derivatives presented above, we note that if $F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}$, then the two derivatives coincide, i.e.,

$$\mathcal{D}_t F = D_t F \text{ for all } F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}.$$

To describe further how \mathcal{D}_t acts on random variables from $L^2(\Omega)$, we state

Theorem 2.3.1. *If $F = \sum_{n=0}^{\infty} I_n(f_n)$ is an element of $\mathbb{D}_{1,2}$, then*

$$\mathcal{D}_t F = \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

As announced previously, we can now mention that δ , sometimes called a divergence operator, can also be defined as the adjoint of the operator \mathcal{D} . So that, if $u \in \text{Dom}(\delta)$, then $\delta(u)$ is an element of $L^2(\Omega)$ characterized by

$$E[F\delta(u)] = E\left[\langle \mathcal{D}F, u \rangle_{L^2([0,T])}\right]$$

for any $F \in \mathbb{D}_{1,2}$.

Finally, we state results that will be used extensively in the subsequent sections.

Theorem 2.3.2 (Chain Rule). *Let $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives (this result can be extended to Lipschitz functions ϕ) and $F = (F_1, \dots, F_m)$ be a random vector such that $F_i \in \mathbb{D}_{1,2}$, $i = 1, \dots, m$. Then $\phi \in \mathbb{D}_{1,2}$ and*

$$\mathcal{D}(\phi(F)) = \sum_{i=1}^m \partial_i \phi(F) \mathcal{D}F_i.$$

Theorem 2.3.3. *Let $u \in \text{Dom}(\delta)$ and $F \in \mathbb{D}_{1,2}$, such that $Fu \in L^2([0, T] \times \Omega)$. Then $Fu \in \text{Dom}(\delta)$ and*

$$\delta(Fu) = F\delta(u) - \langle \mathcal{D}F, u \rangle_{L^2([0, T])},$$

provided that the right side is square integrable.

And, lastly, the most important result.

Theorem 2.3.4 (Integration by Parts). *Let F, G be two random variables such that $F \in \mathbb{D}_{1,2}$ and u be a square integrable random variable such that $\mathcal{D}_u F \neq 0$ a.s. and $Gu(\mathcal{D}_u F)^{-1} \in \text{Dom}(\delta)$. Then, for any continuously differentiable function ϕ with bounded derivative,*

$$E[\phi'(F)G] = E[\phi(F)\delta(Gu(\mathcal{D}_u F)^{-1})].$$

To conclude this introduction we present some more

Examples 3.

(a) If $F = W_{t_1}$, $t_1 \in [0, T]$ fixed, then we can use definition 2 to find the derivative $\mathcal{D}F$. Since $W_{t_1} = \int_0^T \mathbb{1}_{[0, t_1]}(s) dW_s$, then, for any $\gamma(t) = \int_0^T g(s) ds \in H$,

$$\begin{aligned} D_\gamma F &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F(\omega + \varepsilon\gamma) - F(\omega)) = \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\int_0^T \mathbb{1}_{[0, t_1]}(s) [dW_s + \varepsilon d\gamma(s)] - \int_0^T \mathbb{1}_{[0, t_1]}(s) dW_s \right) = \int_0^T \mathbb{1}_{[0, t_1]}(s) g(s) ds. \end{aligned}$$

Hence, for all $t \in [0, T]$,

$$\mathcal{D}_t F = D_t F = \mathbb{1}_{[0, t_1]}(t).$$

(b) If $F = e^{W_{t_1}}$, $t_1 \in [0, T]$ fixed, then, using the previous example and the Chain Rule, for all $t \in [0, T]$,

$$\mathcal{D}_t F = e^{W_{t_1}} \mathcal{D}_t W_{t_1} = e^{W_{t_1}} \mathbb{1}_{[0, t_1]}(t).$$

(c) Let $F = \int_0^T W_{t_1} \delta(W_s)$ with $t_1 \in [0, T]$ fixed. Then the Chain Rule and example 2(b) suggest that, for $t \in [0, T]$,

$$\mathcal{D}_t F = \mathcal{D}_t (W_T W_{t_1} - t_1) = W_{t_1} \mathcal{D}_t W_T + W_T \mathcal{D}_t W_{t_1} = W_{t_1} + W_T \mathbb{1}_{[0, t_1]}(t).$$

Chapter 3

Application of IBP to vanilla, digital and Asian options

In this section some computations related to the Delta, Gamma (second derivative with respect to the initial price) and Vega of vanilla options suggested in [6] are carried out. Since for such vanilla options, whose payoff depends only on terminal price of the underlying asset (stock), the well-known Black-Scholes (BS) formula provides a closed-form solution for the value of the option, formulas for Greeks can be obtained by directly differentiating the solution. On the other hand, it is shown in the book how formulas for these expressions can be obtained using techniques of Malliavin calculus. In what follows Greeks of a European call option are computed by both methods.

The usual setting for the Black-Scholes model is used, i.e. the mean rate of return $\mu_t = \mu$, the volatility $\sigma_t = \sigma$ and the interest rate $r_t = r$ are considered to be constant, so that dynamics of the price of the stock is described by a geometric Brownian motion

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right), \quad t \in [0, T],$$

and price of riskless bond is $B_t = e^{rt}$. Note that by Ito formula this process satisfies a linear

stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Then after introducing discounted stock price and applying Girsanov Theorem we can obtain a measure \mathbf{Q} described by

$$Z_T = \frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\left(-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t\right),$$

so that $\tilde{W}_t = W_t + \frac{\mu-r}{\sigma}t$ is a Brownian motion under \mathbf{Q} and in terms of this new Brownian motion

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right), \quad t \in [0, T].$$

3.1 Computing greeks directly from the BS solution

It is a well-known fact (see, for example, [6]), that in the case of a European call option with strike price K , maturity T , and a continuous and piecewise differentiable payoff $\Phi(x) = (x - K)^+$ which has linear growth, solution to the Black-Scholes equation is

$$\begin{aligned} F(t, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \left(x e^{-\sigma^2/2\tau + \sigma\sqrt{\tau}y} - K e^{-r\tau} \right)^+ dy = \\ &= xN(d_+) - K e^{-r\tau} N(d_-), \end{aligned}$$

where $F(t, S_t)$ is the value of the option at time t , $\tau = T - t$ is the time to maturity and

$$\begin{aligned} d_- &= \frac{\log \frac{x}{K} + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \\ d_+ &= \frac{\log \frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \end{aligned}$$

To clarify computations in what follows,

$$\begin{aligned}\frac{\partial d_-}{\partial x} &= \frac{\partial d_+}{\partial x} = \frac{1}{x} \frac{1}{\sigma \sqrt{\tau}}, \\ \frac{\partial^2 d_-}{\partial x^2} &= \frac{\partial^2 d_+}{\partial x^2} = -\frac{1}{x^2} \frac{1}{\sigma \sqrt{\tau}}, \\ \frac{\partial d_+}{\partial \sigma} &= -\frac{\log \frac{x}{K} + r\tau}{\sigma^2 \sqrt{\tau}} + \frac{\sqrt{\tau}}{2} = -\frac{d_-}{\sigma}, \\ \frac{\partial d_-}{\partial \sigma} &= \frac{\partial d_+}{\partial \sigma} - \sqrt{\tau} = -\frac{d_+}{\sigma}\end{aligned}$$

and, denoting by $N(x)$ and $\phi(x)$ the cdf and pdf of a standard normal distribution, respectively,

$$\begin{aligned}N(x) &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \quad N'(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \\ N''(x) &= \phi'(x) = -\frac{1}{\sqrt{2\pi}} x e^{-x^2/2} = -x\phi(x).\end{aligned}$$

Therefore, at any time $t \in [0, T]$,

1.1)

$$\begin{aligned}\frac{\partial F}{\partial x}(t, x) &= N(d_+) + xN'(d_+) \frac{\partial d_+}{\partial x} - Ke^{-r\tau} N'(d_-) \frac{\partial d_-}{\partial x} = \\ &= N(d_+) + \frac{x\phi(d_+)}{x\sigma\sqrt{\tau}} - \frac{Ke^{-r\tau}\phi(d_-)}{x\sigma\sqrt{\tau}} = \\ &= N(d_+) + \frac{1}{x\sigma\sqrt{\tau}} [x\phi(d_+) - Ke^{-r\tau}\phi(d_-)].\end{aligned}$$

Claim: $Ke^{-r\tau}\phi(d_-) = x\phi(d_+)$. Indeed, since $d_- = d_+ - \sigma\sqrt{\tau}$,

$$\begin{aligned}Ke^{-r\tau}\phi(d_-) &= Ke^{-r\tau}\phi(d_+ - \sigma\sqrt{\tau}) = Ke^{-r\tau} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_+^2 - 2d_+\sigma\sqrt{\tau} + \sigma^2\tau}{2}\right) = \\ &= K\phi(d_+) \exp\left(-r\tau + \log \frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau - \frac{\sigma^2}{2}\tau\right) = x\phi(d_+).\end{aligned}$$

Hence, $\Delta = \frac{\partial F}{\partial S_t}(0, S_0) = N(d_+)$.

1.2)

$$\frac{\partial^2 F}{\partial x^2}(t, x) = N'(d_+) \frac{\partial d_+}{\partial x} = \frac{1}{x\sigma\sqrt{\tau}} N'(d_+),$$

so that $\Gamma = \frac{\partial^2 F}{\partial S_t^2}(0, S_0) = \frac{1}{S_0\sigma\sqrt{T}} N'(d_+)$.

1.3)

$$\begin{aligned} \frac{\partial F}{\partial \sigma}(t, x) &= xN'(d_+) \frac{\partial d_+}{\partial \sigma} - Ke^{-r\tau} N'(d_-) \frac{\partial d_-}{\partial \sigma} = \\ &= -x\phi(d_+) \frac{d_-}{\sigma} + Ke^{-r\tau} \phi(d_-) \frac{d_+}{\sigma}, \end{aligned}$$

but

$$\begin{aligned} Ke^{-r\tau} \phi(d_-) \frac{d_+}{\sigma} &= Ke^{-r\tau} \frac{1}{\sqrt{2\pi}} \left[\frac{d_- + \sigma\sqrt{\tau}}{\sigma} \right] \exp\left(-\frac{(d_- + \sigma\sqrt{\tau})^2}{2}\right) = \\ &= x\phi(d_+) \frac{d_-}{\sigma} + \sqrt{\tau} x\phi(d_+). \end{aligned}$$

Thus, $\frac{\partial F}{\partial \sigma}(t, x) = \sqrt{\tau} x N'(d_+)$ and Vega is $v = \frac{\partial F}{\partial \sigma}(0, S_0) = \sqrt{T} S_0 N'(d_+)$.

3.2 Computing greeks using IBP for vanilla options

Next, we derive formulas for Greeks of European options using the integration by parts formula of Malliavin Calculus. Suppose that the payoff depends only on the terminal price of the stock, $\Phi = \Phi(S_T)$, where Φ is a Lipschitz function.

Delta. Since

$$\begin{aligned} \frac{\partial S_T}{\partial S_0} &= \frac{\partial}{\partial S_0} S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) = \frac{S_T}{S_0}, \\ \Delta &= \frac{\partial}{\partial S_0} E_{\mathbf{Q}}[e^{-rT} \Phi(S_T)] = \frac{e^{-rT}}{S_0} E_{\mathbf{Q}}[\Phi'(S_T) S_T]. \end{aligned}$$

Now IBP formula can be applied with $u = 1$ and $F = G = S_T$. Since, by the Chain rule,

$$\mathcal{D}_t S_T = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) T + \sigma W_T \right) \sigma \mathcal{D}_t W_T = S_T \sigma \mathbb{1}_{[0,T]}(t) = \sigma S_T,$$

$$\mathcal{D}^u S_T = \langle \mathcal{D} S_T, u \rangle = \int_0^T \mathcal{D}_t S_T dt = \int_0^T \sigma S_T dt = \sigma T S_T.$$

Then, since Skorohod integral of an adapted process is equal to the Ito integral,

$$\delta \left(Gu(\mathcal{D}^u F)^{-1} \right) = \delta \left(S_T \left(\int_0^T \mathcal{D}_t S_T dt \right)^{-1} \right) = \delta \left(\frac{1}{\sigma T} \right) = \int_0^T \frac{1}{\sigma T} dW_t = \frac{W_T}{\sigma T},$$

so that

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} E_{\mathbf{Q}} [\Phi(S_T) W_T].$$

Gamma.

$$\begin{aligned} \Gamma &= \frac{\partial}{\partial S_0} e^{-rT} E_{\mathbf{Q}} \left[\Phi'(S_T) \frac{\partial S_T}{\partial S_0} \right] = e^{-rT} E_{\mathbf{Q}} \left[\Phi''(S_T) \left(\frac{\partial S_T}{\partial S_0} \right)^2 \right] = \\ &= \frac{e^{-rT}}{S_0^2} E_{\mathbf{Q}} [\Phi''(S_T) S_T^2]. \end{aligned}$$

Here we can apply IBP formula with $u = 1$, $F = S_t$ and $G = S_T^2$, and Theorem 2.3.3 to get

$$\begin{aligned} \delta \left(Gu(\mathcal{D}^u F)^{-1} \right) &= \delta \left(\frac{S_T}{\sigma T} \right) = \frac{S_T}{\sigma T} \delta(1) - \int_0^T \mathcal{D}_t \frac{S_T}{\sigma T} dt = \\ &= \frac{S_T}{\sigma T} \int_0^T dW_t - \frac{1}{\sigma T} S_T \sigma T = S_T \left(\frac{W_T}{\sigma T} - 1 \right), \end{aligned}$$

so that

$$E_{\mathbf{Q}} [\Phi''(S_T) S_T^2] = E_{\mathbf{Q}} \left[\Phi'(S_T) S_T \left(\frac{W_T}{\sigma T} - 1 \right) \right].$$

Since the derivative of the payoff function is still present, we apply IBP another time with

$u = 1$, $F = S_T$ and $G = S_T \left(\frac{W_T}{\sigma T} - 1 \right)$. This time,

$$\begin{aligned} & \delta \left(S_T \left(\frac{W_T}{\sigma T} - 1 \right) \left(\int_0^T \mathcal{D}_t S_T dt \right)^{-1} \right) = \delta \left(\frac{W_T}{\sigma^2 T^2} - \frac{1}{\sigma T} \right) = \\ & = \delta \left(\frac{W_T}{\sigma^2 T^2} \right) - \delta \left(\frac{1}{\sigma T} \right) = \frac{W_T}{\sigma^2 T^2} \delta(1) - \int_0^T \mathcal{D}_t \frac{W_T}{\sigma^2 T^2} dt - \frac{W_T}{\sigma T} = \\ & = \frac{W_T^2}{\sigma^2 T^2} - \frac{1}{\sigma^2 T} - \frac{W_T}{\sigma T}, \end{aligned}$$

so that

$$E_{\mathbf{Q}} \left[\Phi'(S_T) S_T \left(\frac{W_T}{\sigma T} - 1 \right) \right] = E_{\mathbf{Q}} \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma^2 T^2} - \frac{1}{\sigma^2 T} - \frac{W_T}{\sigma T} \right) \right].$$

Therefore,

$$\Gamma = \frac{e^{-rT}}{S_0^2 \sigma T} E_{\mathbf{Q}} \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right].$$

Vega. Noticing that

$$v = \frac{\partial}{\partial \sigma} E_{\mathbf{Q}} [e^{-rT} \Phi(S_T)] = e^{-rT} E_{\mathbf{Q}} [\Phi'(S_T) S_T (W_T - \sigma T)] = S_0^2 \sigma T \Gamma,$$

we have

$$v = e^{-rT} E_{\mathbf{Q}} \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right].$$

Returning to the case of a European call option, we now compute the greeks using latter formulas.

2.1) Delta.

To make computations simpler we replace W_T by $-W_T$, so that

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} E_{\mathbf{Q}} [\Phi(S_T) (-W_T)].$$

First, notice that

$$e^{-rT} \Phi(S_T) = 0 \quad \text{if} \quad S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t - \sigma W_t\right) \leq e^{-rT} K,$$

which results in

$$W_T \geq \frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma} := d_-^*.$$

Then

$$\begin{aligned} e^{-rT} E_{\mathbf{Q}}[\Phi(S_T) W_T] &= E_{\mathbf{Q}} \left[\left(S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma W_T\right) - e^{-rT} K \right)^+ W_T \right] = \\ &= \int_{-\infty}^{d_-^*} \left[S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma y\right) - e^{-rT} K \right] \frac{y}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2T}\right) dy = \\ &= S_0 \int_{-\infty}^{d_-^*} \frac{y}{\sqrt{2\pi T}} \exp\left(-\left(\frac{y^2}{2T} + \sigma y + \frac{\sigma^2 T}{2}\right)\right) dy - \\ &\quad - e^{-rT} K \int_{-\infty}^{d_-^*} \frac{y}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2T}\right) dy. \end{aligned}$$

Computing each of the integrals, we have

$$\Delta = N(d_+) + \frac{1}{S_0 \sigma \sqrt{T}} [S_0 \phi(d_+) - e^{-rT} K \phi(d_-)] = N(d_+),$$

where the second term is zero by the claim from **1.1**).

2.2) Gamma.

$$\begin{aligned} \Gamma &= \frac{e^{-rT}}{S_0^2 \sigma T} E_{\mathbf{Q}} \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} + W_T \right) \right] = \\ &= \frac{e^{-rT}}{S_0^2 \sigma^2 T^2} \int_{-\infty}^{d_-^*} \left[S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma y\right) - e^{-rT} K \right] \frac{y^2}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2T}\right) dy - \end{aligned}$$

$$\begin{aligned}
& -\frac{e^{-rT}}{S_0^2\sigma^2T} \int_{-\infty}^{d_-^*} \left[S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma y\right) - e^{-rT}K \right] \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2T}\right) dy + \\
& + \frac{e^{-rT}}{S_0^2\sigma T} \int_{-\infty}^{d_-^*} \left[S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T - \sigma y\right) - e^{-rT}K \right] \frac{y}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2T}\right) dy.
\end{aligned}$$

Computing three latter integrals and summing the results, we obtain

$$\Gamma = \frac{1}{S_0\sigma^2T}N''(d_+) + \frac{2}{S_0\sigma\sqrt{T}}N'(d_+) - \frac{Ke^{-rT}}{S_0^2\sigma^2T}N''(d_-) = \frac{1}{S_0\sigma\sqrt{T}}N'(d_+).$$

2.3) Vega. Since $v = S_0^2\sigma T\Gamma$,

$$v = e^{-rT}E_{\mathbf{Q}}\left[\Phi(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} + W_T\right)\right] = \sqrt{T}S_0N'(d_+).$$

3.3 Computing greeks using IBP for digital options

Here, we perform similar computations for a particular digital option with payoff $\Phi(x) = \mathbb{1}_{\{x>K\}}$.

Solution of the Black-Scholes equation then takes form

$$\begin{aligned}
F(t,x) &= e^{-r\tau} \int_{-\infty}^{\infty} \Phi\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}y}\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \\
&= e^{-r\tau} \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = e^{-r\tau}N(d_-).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial F}{\partial x}(t,x) &= e^{-r\tau}N'(d_-) \frac{\partial d_-}{\partial x} = \frac{e^{-r\tau}}{x\sigma\sqrt{\tau}}\phi(d_-), \\
\frac{\partial^2 F}{\partial x^2}(t,x) &= e^{-r\tau}N''(d_-) \left(\frac{\partial d_-}{\partial x}\right)^2 + e^{-r\tau}N'(d_-) \frac{\partial^2 d_-}{\partial x^2} = \\
&= -\frac{e^{-r\tau}}{x^2\sigma\sqrt{\tau}} \left[\frac{1}{\sigma\sqrt{\tau}}d_- + 1\right] \phi(d_-) = -\frac{e^{-r\tau}}{x^2\sigma^2\tau}d_+\phi(d_-)
\end{aligned}$$

and

$$\frac{\partial F}{\partial \sigma}(t, x) = e^{-r\tau} N'(d_-) \frac{\partial d_-}{\partial \sigma} = -\frac{e^{-r\tau}}{\sigma} \phi(d_-).$$

Therefore,

$$\begin{aligned}\Delta &= \frac{\partial F}{\partial x}(0, S_0) = \frac{e^{-rT}}{S_0 \sigma \sqrt{T}} \phi(d_-), \\ \Gamma &= \frac{\partial^2 F}{\partial x^2}(0, S_0) = -\frac{e^{-rT}}{S_0^2 \sigma^2 T} d_+ \phi(d_-)\end{aligned}$$

and

$$v = \frac{\partial F}{\partial \sigma}(0, S_0) = -\frac{e^{-rT}}{\sigma} d_+ \phi(d_-).$$

On the other hand, using formulas obtained with Malliavin calculus,

3.1) Delta.

$$\begin{aligned}\Delta &= \frac{e^{-rT}}{S_0 \sigma T} E_{\mathbf{Q}}[\Phi(S_T) W_T] = \frac{e^{-rT}}{S_0 \sigma T} \int_{-\infty}^{\infty} \mathbb{1}_{S_T > K}(y) \frac{y}{\sqrt{2\pi T}} e^{-y^2/2T} dy = \\ &= \frac{e^{-rT}}{S_0 \sigma T} \int_{-\infty}^{d_-^*} \frac{y}{\sqrt{2\pi T}} e^{-y^2/2T} dy = \frac{e^{-rT}}{S_0 \sigma \sqrt{T}} \phi(d_-).\end{aligned}$$

3.2) Gamma.

$$\begin{aligned}\Gamma &= \frac{e^{-rT}}{S_0^2 \sigma T} E_{\mathbf{Q}} \left[\Phi(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} + W_T \right) \right] = \\ &= \frac{e^{-rT}}{S_0^2 \sigma^2 T^2} \int_{-\infty}^{d_-^*} \frac{y^2}{\sqrt{2\pi T}} e^{-y^2/2T} dy - \\ &= -\frac{e^{-rT}}{S_0^2 \sigma^2 T} \int_{-\infty}^{d_-^*} \frac{1}{\sqrt{2\pi T}} e^{-y^2/2T} dy + \\ &\quad + \frac{e^{-rT}}{S_0^2 \sigma T} \int_{-\infty}^{d_-^*} \frac{y}{\sqrt{2\pi T}} e^{-y^2/2T} dy =\end{aligned}$$

$$= \frac{e^{-rT}}{S_0^2 \sigma \sqrt{T}} \left(\frac{1}{\sigma \sqrt{T}} N''(d_-) - N'(d_-) \right) = -\frac{e^{-rT}}{S_0^2 \sigma^2 T} d_+ \phi(d_-).$$

3.3) *Vega*.

$$v = S_0^2 \sigma T \Gamma = -\frac{e^{-rT}}{\sigma} d_+ \phi(d_-).$$

3.4 Computing greeks using IBP for Asian options

An asian option is an exotic option whose payoff is determined by the average price of the underlying stock over some pre-set period of time. For example, an asian call option with strike price K has the payoff

$$\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) = \max \left(\frac{1}{T} \int_0^T S_t dt - K, 0 \right).$$

In such a situation the price of the option is not known in closed form. But, since the price of this option at time $t = 0$ is given by

$$V_0 = e^{-rT} E_Q \left[\Phi \left(\frac{1}{T} \int_0^T S_t dt \right) \right],$$

we can still find delta and gamma using the IBP formula.

4.1) If we denote $\bar{S}_T = 1/T \int_0^T S_t dt$, delta for this type of options is

$$\Delta = \frac{\partial V_0}{\partial S_0} = e^{-rT} E_Q \left[\Phi'(\bar{S}_T) \frac{\partial \bar{S}_T}{\partial S_0} \right] = \frac{e^{-rT}}{S_0} E_Q [\Phi'(\bar{S}_T) \bar{S}_T].$$

We can now apply IBP with $G = F = \bar{S}_T$ and $u_t = S_t$. For this, we first compute

$$\mathcal{D}_t F = \frac{1}{T} \int_0^T \mathcal{D}_t S_r dr = \frac{1}{T} \int_0^T S_r \sigma \mathbb{1}_{[t, T]}(r) dr = \frac{\sigma}{T} \int_t^T S_r dr$$

and, changing the order of integration and renaming variables,

$$\begin{aligned}\int_0^T S_t \mathcal{D}_t F dt &= \frac{\sigma}{T} \int_0^T S_t \left(\int_t^T S_r dr \right) dt = \frac{\sigma}{T} \int_0^T S_r \left(\int_0^r S_t dt \right) dr = \\ &= \frac{\sigma}{T} \int_0^T S_t \left(\int_0^t S_r dr \right) dt,\end{aligned}$$

so that

$$\int_0^T S_t \mathcal{D}_t F dt = \frac{\sigma}{2T} \left(\int_0^T S_t dt \right)^2.$$

Further, we compute

$$\delta \left(Gu(D^u F)^{-1} \right) = \delta \left(\frac{\bar{S}_T S.}{\int_0^T S_t \mathcal{D}_t F dt} \right) = \frac{2}{\sigma} \delta \left(\frac{S.}{\int_0^T S_t dt} \right),$$

for which, after applying Theorem 2.3.3, we obtain

$$\begin{aligned}& \frac{2}{\sigma} \left(\frac{\delta(S.)}{\int_0^T S_t dt} - \left\langle \mathcal{D} \left(\int_0^T S_t dt \right)^{-1}, S. \right\rangle \right) = \\ &= \frac{2}{\sigma} \left(\frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + \frac{\sigma \int_0^T S_t \left(\int_t^T S_r dr \right) dt}{\left(\int_0^T S_t dt \right)^2} \right) = \frac{2}{\sigma} \left(\frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + \frac{\sigma}{2} \right) = \\ &= \frac{2}{\sigma} \frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + 1.\end{aligned}$$

Finally, since $\int_0^T S_t dW_t = 1/\sigma(S_T - S_0 - \int_0^T r S_t dt)$,

$$\delta \left(Gu(D^u F)^{-1} \right) = \frac{2}{\sigma^2} \left(\frac{S_T - S_0}{T \bar{S}_T} - (r - \sigma^2/2) \right)$$

and, thus, we have

$$\Delta = \frac{2e^{-rT}}{S_0 \sigma^2} E_Q \left[\Phi(\bar{S}_T) \left(\frac{S_T - S_0}{T \bar{S}_T} - (r - \sigma^2/2) \right) \right].$$

4.2) We now perform similar steps to compute gamma of an asian option. We start with apply-

ing the IBP formula to

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = e^{-rT} E_Q \left[\Phi''(\bar{S}_T) \left(\frac{\bar{S}_T}{S_0} \right)^2 \right] = \frac{e^{-rT}}{S_0^2} E_Q \left[\Phi''(\bar{S}_T) \bar{S}_T^2 \right]$$

with $F = \bar{S}_T$, $G = \bar{S}_T^2$ and $u_t = S_t$. Using results obtained in calculating delta,

$$\begin{aligned} \delta(Gu(D^u F)^{-1}) &= \delta \left(\frac{\bar{S}_T^2 S.}{\int_0^T S_t \mathcal{D}_t F dt} \right) = \delta \left(\frac{\bar{S}_T^2 S.}{\frac{\sigma}{2T} \left(\int_0^T S_t dt \right)^2} \right) = \\ &= \frac{2}{\sigma T} \delta(S.) = \frac{2}{\sigma T} \int_0^T S_t dW_t = \frac{2}{\sigma^2 T} \left(S_T - S_0 - \int_0^T r S_t dt \right) = \\ &= \frac{2}{\sigma^2 T} (S_T - S_0) - \frac{2r}{\sigma^2} \bar{S}_T, \end{aligned}$$

so that

$$\Gamma = \frac{2e^{-rT}}{S_0^2 \sigma^2 T} E_Q [\Phi'(\bar{S}_T) (S_T - S_0)] - \frac{2re^{-rT}}{S_0^2 \sigma^2} E_Q [\Phi'(\bar{S}_T) \bar{S}_T].$$

Let us denote the first and second expectations from the previous expression by A_1 and A_2 , respectively. Then A_2 was obtained in the previous computation for delta and to find A_1 we can use IBP with $F = \bar{S}_T$, $G = S_T - S_0$ and $u_t = S_t$. Using Theorem 2.3.3 again, we get

$$\begin{aligned} \delta(Gu(D^u F)^{-1}) &= \delta \left(\frac{(S_T - S_0) S.}{\int_0^T S_t \mathcal{D}_t F dt} \right) = \delta \left(\frac{(S_T - S_0) S.}{\frac{\sigma}{2T} \left(\int_0^T S_t dt \right)^2} \right) = \\ &= \frac{2}{\sigma T} \delta \left(\frac{(S_T - S_0) S.}{\bar{S}_T^2} \right) = \frac{2}{\sigma T} \left(\frac{S_T - S_0}{\bar{S}_T^2} \delta(S.) - \left\langle \mathcal{D} \frac{S_T - S_0}{\bar{S}_T^2}, S. \right\rangle \right) = \\ &= \frac{2}{\sigma T} (B_1 - B_2). \end{aligned}$$

Here,

$$B_1 = \frac{S_T - S_0}{\bar{S}_T^2} \delta(S.) = \frac{S_T - S_0}{\bar{S}_T^2} \int_0^T S_t dW_t =$$

$$= \frac{S_T - S_0}{\bar{S}_T^2} \frac{1}{\sigma} \left(S_T - S_0 - \int_0^T r S_t dt \right) = \frac{1}{\sigma} \frac{(S_T - S_0)^2}{\bar{S}_T^2} - \frac{rT}{\sigma} \frac{S_T - S_0}{\bar{S}_T}.$$

Further, since $S_T - S_0$ and \bar{S}_T^{-2} are smooth random variables, we can use Malliavin version of a product rule, a consequence of Theorem 2.3.1, to find the derivative in B_2 . Since

$$\mathcal{D}_t (S_T - S_0) = \sigma S_T$$

and

$$\mathcal{D}_t \bar{S}_T^{-2} = -2\bar{S}_T^{-3} \mathcal{D}_t \bar{S}_T = -\frac{2\sigma}{T} \frac{\int_t^T S_r dr}{\bar{S}_T^3},$$

we have

$$\begin{aligned} \mathcal{D}_t \frac{S_T - S_0}{\bar{S}_T^2} &= [\mathcal{D}_t (S_T - S_0)] \bar{S}_T^{-2} + (S_T - S_0) [\mathcal{D}_t \bar{S}_T^{-2}] = \\ &= \sigma \frac{S_T}{\bar{S}_T^2} - \frac{2\sigma}{T} \frac{(S_T - S_0)}{\bar{S}_T^3} \int_t^T S_r dr. \end{aligned}$$

Thus,

$$\begin{aligned} B_2 &= \left\langle \mathcal{D} \frac{S_T - S_0}{\bar{S}_T^2}, S \right\rangle = \sigma \frac{S_T}{\bar{S}_T^2} \int_0^T S_t dt - \frac{2\sigma}{T} \frac{(S_T - S_0)}{\bar{S}_T^3} \int_0^T S_t \left(\int_t^T S_r dr \right) dt = \\ &= \sigma T \frac{S_T}{\bar{S}_T} - \sigma T \frac{(S_T - S_0)}{\bar{S}_T} = \sigma T \frac{S_0}{\bar{S}_T}, \end{aligned}$$

so that, applying the IBP formula,

$$\begin{aligned} A_1 &= E_Q [\Phi'(\bar{S}_T) (S_T - S_0)] = E_Q [\Phi(\bar{S}_T) \delta(Gu(D^\mu F)^{-1})] = \\ &= \frac{2}{\sigma T} E_Q [\Phi(\bar{S}_T) (B_1 - B_2)] = \\ &= \frac{2}{\sigma^2} E_Q \left[\Phi(\bar{S}_T) \left(\frac{1}{T} \frac{(S_T - S_0)^2}{\bar{S}_T^2} - r \frac{S_T - S_0}{\bar{S}_T} - \sigma^2 \frac{S_0}{\bar{S}_T} \right) \right]. \end{aligned}$$

Plugging all these results back into the gamma, we obtain

$$\begin{aligned}\Gamma &= \frac{2e^{-rT}}{S_0^2\sigma^2T}A_1 - \frac{2re^{-rT}}{S_0^2\sigma^2}A_2 = \\ &= \frac{4e^{-rT}}{S_0^2\sigma^4T}E_Q \left[\Phi(\bar{S}_T) \left(\frac{1}{T} \frac{(S_T - S_0)^2}{\bar{S}_T^2} - 2r \frac{S_T - S_0}{\bar{S}_T} - \sigma^2 \frac{S_0}{\bar{S}_T} + rT(r - \sigma^2/2) \right) \right].\end{aligned}$$

Chapter 4

Computing Delta and Gamma using IBP for Barrier and Lookback Options

In this section we investigate a technique presented in the article [3] for computation of delta and gamma for a collection of various barrier and lookback options and perform some explicit computations based on the results. The general form of the payoff function for options in this group is considered to be of the form $\Phi(\max_{s \leq t, s \in I} S_s, \min_{s \leq t, s \in I} S_s, S_T)$, where $I \subset [0, T]$. A significant difference from the previous situations is that the IBP cannot be applied directly due to the fact that minima and maxima of processes considered further are not twice differentiable. To overcome this, E. Gobet and A. Kohatsu-Higa offered to use a localization procedure by introducing a dominating process.

To simplify notation of this chapter, separately from chapter 3 we consider another stock whose price dynamics is assumed to satisfy

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where constants r and σ are again a riskless rate of return and a volatility of the stock, and W_t is a Brownian motion under some risk-neutral probability \mathbf{P} . To make further computations simpler, we'll consider a logarithm transformation of S_t , $X_t = A(S_t)$, where $A(y) = \ln(y)/\sigma$.

Then, since the quadratic variation of S_t is

$$\langle S \rangle_t = \int_0^t \sigma^2 S_s^2 ds,$$

applying Ito formula,

$$\begin{aligned} A(S_t) &= A(S_0) + \int_0^t A'(S_s) dS_s + \frac{1}{2} \int_0^t A''(S_s) d\langle S \rangle_s = \\ &= A(S_0) + \int_0^t \frac{1}{\sigma S_s} r S_s ds + \int_0^t \frac{1}{\sigma S_s} \sigma S_s dW_s - \frac{1}{2} \int_0^t \frac{1}{\sigma S_s^2} \sigma^2 S_s^2 ds = \\ &= A(S_0) + \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right) t + W_t. \end{aligned}$$

Denoting the transformed initial price by $x = A(S_0)$, we obtain that

$$X_t = x + \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right) t + W_t.$$

A number of technical assumptions are required.

1. It is assumed that there exists a non-decreasing adapted right-continuous process $(Y_t)_{0 \leq t \leq T}$ that dominates X :

$$|X_t - x| \leq Y_t \quad \text{for all } t \in I.$$

Two such processes presented in the paper will be mentioned later.

2. For the general form of a payoff function it is assumed that there exists $a > 0$ such that the function $\Phi(M, m, z)$ does not depend on the variables M and m for any (M, m, z) such that $0 \leq M - x < a$ or $0 \leq x - m < a$. This support condition is satisfied by most usual barrier and lookback options and can be simplified in cases when the payoff Φ depends only on one from the pair M, m .

3. There exists a positive function $\alpha : \mathbf{N} \rightarrow \mathbf{R}^+$, with $\lim_{q \rightarrow \infty} \alpha(q) = \infty$, such that for any $q \geq 1$

$$E[Y_t^q] \leq C_q t^{\alpha(q)} \quad \text{for all } t \in [0, T].$$

4. Let $\Psi : [0, \infty) \rightarrow [0, 1]$ be a C_b^∞ function with

$$\Psi(x) = \begin{cases} 1 & \text{if } x \leq a/2, \\ 0 & \text{if } x \geq a, \end{cases}$$

where the number a is the one appearing in the assumption 2. Then for $q \in \mathbf{N}$ it is assumed that the random variable $\Psi(Y_t)$ belongs to $\mathbb{D}_{q,\infty}$ (a generalization of $\mathbb{D}_{1,2}$, see [6]) for each t , and that for all $j = 1, \dots, q$,

$$\sup_{r_1, \dots, r_j \in [0, T]} E \left[\sup_{r_1 \vee \dots \vee r_j \leq t \leq T} |\mathcal{D}_{r_1, \dots, r_j} \Psi(Y_t)|^p \right] \leq C_p \quad \text{for all } p \geq 1.$$

Moreover, for $q \geq 2$, the $q-1$ first derivatives of $\Psi(Y_t)$ with respect to x , $\partial_x(\Psi(Y_t)), \dots, \partial_x^{q-1}(\Psi(Y_t))$, exist and satisfy the same estimates as above.

We now introduce some new notation: let $M_T = \max_{s \leq T, s \in I} X_s$, $m_T = \min_{s \leq T, s \in I} X_s$ and

$$\tilde{\Phi}(u_1, u_2, u_3) = \Phi(A^{-1}(u_1), A^{-1}(u_2), A^{-1}(u_3)) = \Phi(e^{\sigma u_1}, e^{\sigma u_2}, e^{\sigma u_3}).$$

Since X_t is already a Brownian motion with deterministic drift under probability \mathbf{P} , no change of measure is needed and the following lemma can be applied directly. Before we proceed to the lemma, notice that

$$E_{\mathbf{P}} \left[\Phi \left(\max_{s \leq t, s \in I} S_s, \min_{s \leq t, s \in I} S_s, S_t \right) \right] = E_{\mathbf{P}} \left[\tilde{\Phi}(M_T, m_T, X_T) \right].$$

Lemma 4.0.1. *Let V be a standard Brownian motion and $V_t^f = V_t + \int_0^t f(s)ds$ for a deterministic*

function f . Then random variables $\max_{s \leq T, s \in I} V_s^f$ and $\min_{s \leq T, s \in I} V_s^f$ belong to $\mathbb{D}_{1,\infty}$ and their first weak derivatives are

$$\mathcal{D}_t \left(\max_{s \leq T, s \in I} V_s^f \right) = \mathbb{1}_{t \leq \tau^M}, \quad \mathcal{D}_t \left(\min_{s \leq T, s \in I} V_s^f \right) = \mathbb{1}_{t \leq \tau^m} \quad \text{for } t \in [0, T],$$

where τ^M and τ^m are random times in $I \cap [0, T]$ such that

$$V_{\tau^M}^f = \max_{s \leq T, s \in I} V_s^f \quad \text{and} \quad V_{\tau^m}^f = \min_{s \leq T, s \in I} V_s^f.$$

Theorem 4.0.2. Suppose that assumptions 1, 2 and 3 hold and that payoff function Φ is smooth with bounded derivatives. Then

1) if also Y satisfies assumption 4 with $q = 1$, then

$$\Delta = \frac{1}{\sigma S_0} E_{\mathbf{P}} \left[\tilde{\Phi}(M_T, m_T, X_T) H_1 \right],$$

where

$$H_1 = \delta \left(\frac{1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right);$$

2) if also Y satisfies assumption 4 with $q = 2$, then

$$\Gamma = \frac{1}{\sigma^2 S_0^2} E_{\mathbf{P}} \left[\tilde{\Phi}(M_T, m_T, X_T) H_2 \right]$$

where

$$H_2 = \delta \left(\frac{H_1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right) + \partial_x H_1.$$

Proof. 1) Differentiating with respect to the initial price S_0 , since $\partial_{S_0} = \frac{1}{\sigma S_0} \partial_x$, we obtain

$$\Delta = \partial_{S_0} E_{\mathbf{P}} \left[\Phi \left(\max_{s \leq t, s \in I} S_s, \min_{s \leq t, s \in I} S_s, S_T \right) \right] = \frac{1}{\sigma S_0} E_{\mathbf{P}} \left[\partial_x \tilde{\Phi}(M_T, m_T, X_T) \right] =$$

$$= \frac{1}{\sigma S_0} E \mathbf{P} \left[\left(\tilde{\Phi}'_M + \tilde{\Phi}'_m + \tilde{\Phi}'_z \right) (M_T, m_T, X_T) \right].$$

On the other hand, using the Chain Rule and Lemma 4.0.1, the Malliavin derivative of the transformed payoff function is

$$\begin{aligned} \mathcal{D}_t \tilde{\Phi} &= \tilde{\Phi}'_M \mathcal{D}_t M_T + \tilde{\Phi}'_m \mathcal{D}_t m_T + \tilde{\Phi}'_z \mathcal{D}_t X_T = \\ &= \tilde{\Phi}'_M \mathbb{1}_{t \leq \tau^M} + \tilde{\Phi}'_m \mathbb{1}_{t \leq \tau^m} + \tilde{\Phi}'_z, \end{aligned} \quad (*)$$

since $\mathcal{D}_t X_T = \mathbb{1}_{[0, T]}(t) = 1$.

The idea now is to try and connect the expression in the expectation with a Malliavin derivative of some random variable. So, we want to show that with the smoothing function Ψ from assumption 4 we get

$$\mathcal{D}_t \tilde{\Phi} \Psi(Y_t) = (\operatorname{div} \tilde{\Phi})(M_T, m_T, X_T) \Psi(Y_t),$$

where $\operatorname{div} \tilde{\Phi} = \tilde{\Phi}'_M + \tilde{\Phi}'_m + \tilde{\Phi}'_z$. To prove it we consider events

$$A = \left\{ \max_{s \leq T, s \in I} X_s - x \leq a \right\} \cup \left\{ \min_{s \leq T, s \in I} X_s - x \geq -a \right\} \quad \text{and}$$

$$A^c = \left\{ \max_{s \leq T, s \in I} X_s - x > a \right\} \cap \left\{ \min_{s \leq T, s \in I} X_s - x < -a \right\}.$$

On A , the assumption 2 implies that $\tilde{\Phi}'_M = \tilde{\Phi}'_m = 0$, so that both sides of the equation (*) are equal to $\tilde{\Phi}'_z \Psi(Y_t)$.

On A^c , if $\Psi(Y_t) \neq 0$, we have $Y_t < a$, which implies that $-a < X_t - x < a$, while $\max_{s \leq T, s \in I} X_s - x > a$. The last two inequalities mean that, for such t , $t < \tau^M$ and, hence, $\mathbb{1}_{t \leq \tau^M} \Psi(Y_t) = \Psi(Y_t)$. Similarly, if $\Psi(Y_t) \neq 0$, $\min_{s \leq T, s \in I} X_s - x < -a$ implies that $t < \tau^m$ and $\mathbb{1}_{t \leq \tau^m} \Psi(Y_t) = \Psi(Y_t)$. Thus, (*) holds on A^c , so, it is proved.

Then integrating both sides of (*) from 0 to T , we get

$$(\operatorname{div} \tilde{\Phi}) \int_0^T \Psi(Y_t) dt = \int_0^T \mathcal{D}_t \tilde{\Phi} \Psi(Y_t) dt$$

or

$$\begin{aligned} & (\operatorname{div} \tilde{\Phi})(M_T, m_T, X_T) = \\ &= \frac{1}{\int_0^T \Psi(Y_t) dt} \int_0^T \mathcal{D}_t \tilde{\Phi} \Psi(Y_t) dt = \left\langle \mathcal{D} \tilde{\Phi}, \frac{1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right\rangle. \end{aligned}$$

Now, since $\Psi(Y_t)$ is smooth and satisfies assumptions 3 and 4 with $q = 1$, all technical conditions of the integration by parts formula are satisfied and it yields

$$\begin{aligned} & E_{\mathbf{P}} \left[(\operatorname{div} \tilde{\Phi})(M_T, m_T, X_T) \right] = \\ &= E_{\mathbf{P}} \left[\left\langle \mathcal{D} \tilde{\Phi}, \frac{1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right\rangle \right] = E_{\mathbf{P}} \left[\tilde{\Phi} \delta \left(\frac{1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right) \right]. \end{aligned}$$

Hence,

$$\Delta = \frac{1}{\sigma S_0} E_{\mathbf{P}} \left[\tilde{\Phi} \delta \left(\frac{1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right) \right].$$

2) Similarly,

$$\begin{aligned} \Gamma &= \partial_{S_0 S_0}^2 E_{\mathbf{P}} \left[\Phi \left(\max_{s \leq T, s \in I} S_s, \min_{s \leq T, s \in I} S_s, S_T \right) \right] = \frac{1}{\sigma^2 S_0^2} E_{\mathbf{P}} \left[\partial_x \tilde{\Phi}(M_T, m_T, X_T) H_1 \right] = \\ &= \frac{1}{\sigma^2 S_0^2} E_{\mathbf{P}} \left[\left(\tilde{\Phi}'_M + \tilde{\Phi}'_m + \tilde{\Phi}'_z \right) (M_T, m_T, X_T) H_1 + \tilde{\Phi}(M_T, m_T, X_T) \partial_x H_1 \right]. \end{aligned}$$

Since H_1 is again a smooth random variable, we can repeat the proof of the previous part to obtain

$$\frac{1}{\sigma^2 S_0^2} \Gamma = E_{\mathbf{Q}} \left[\tilde{\Phi} \left\{ \delta \left(\frac{H_1}{\int_0^T \Psi(Y_t) dt} \Psi(Y) \right) + \partial_x H_1 \right\} \right] \quad QED.$$

As both expressions for delta and gamma depend on the dominating process Y_t , we can state two such processes offered by the authors and proceed a little bit further in computations. Thus,

using the Theorems 2.2.1 and 2.3.3,

$$\begin{aligned}
H_1 &= \delta \left(\frac{1}{\int_0^T \Psi(Y_t) dt} \Psi(Y_\cdot) \right) = \\
&= \frac{1}{\int_0^T \Psi(Y_t) dt} \delta(\Psi(Y_\cdot)) - \int_0^T \mathcal{D}_s \left(\frac{1}{\int_0^T \Psi(Y_t) dt} \right) \Psi(Y_s) ds = \\
&= \frac{\int_0^T \Psi(Y_t) dW_t}{\int_0^T \Psi(Y_t) dt} - \int_0^T \mathcal{D}_s \left(\frac{1}{\int_0^T \Psi(Y_t) dt} \right) \Psi(Y_s) ds,
\end{aligned}$$

where, by the Chain Rule,

$$\mathcal{D}_s \frac{1}{\int_0^T \Psi(Y_t) dt} = - \frac{1}{\left(\int_0^T \Psi(Y_t) dt \right)^2} \int_0^T \Psi'(Y_t) \mathcal{D}_s Y_t \mathbb{1}_{[s, T]}(t) dt.$$

Hence,

$$H_1 = \frac{\int_0^T \Psi(Y_t) dW_t}{\int_0^T \Psi(Y_t) dt} + \frac{1}{\left(\int_0^T \Psi(Y_t) dt \right)^2} \int_0^T \left(\int_s^T \Psi'(Y_t) \mathcal{D}_s Y_t dt \right) \Psi(Y_s) ds.$$

Now, the first and simplest dominating process is $Y_t = \max_{s \leq t} (X_s - x) - \min_{s \leq t} (X_s - x)$. This process satisfies assumptions required for computing delta, but, as was mentioned before, having extrema of a Brownian motion in it, is not smooth enough for computing gamma. Here, using Lemma 4.0.1, we can find that

$$\mathcal{D}_s Y_t = \mathbb{1}_{s \leq \tau^{M_t}} - \mathbb{1}_{s \leq \tau^{m_t}},$$

where τ^{M_t} and τ^{m_t} are random times such that $X_{\tau^{M_t}} = \max_{s \leq t} X_s$ and $X_{\tau^{m_t}} = \min_{s \leq t} X_s$, respectively.

The second process is the averaged modulus continuity process and it is defined, for an even γ , as

$$Y_t = 8t^{m/\gamma} \frac{m+2}{m} \left(4 \int_0^t \int_0^t \frac{|X_\tau - X_u|^\gamma}{|\tau - u|^{m+2}} d\tau du \right)^{1/\gamma}.$$

In the case when $0 < m < \frac{\gamma}{2} - 2$, it can be proved that it is a non-decreasing, adapted, continuous process satisfying the conditions required for obtaining gamma. For such a process, applying the

Chain Rule several times,

$$\begin{aligned}
\mathcal{D}_s Y_t &= 8 \cdot 4^{1/\gamma} \frac{m+2}{\gamma m} t^{m/\gamma} \left(\int_0^t \int_0^t \frac{|X_\tau - X_u|^\gamma}{|\tau - u|^{m+2}} d\tau du \right)^{(1-\gamma)/\gamma} \mathcal{D}_s \left(\int_0^t \int_0^t \frac{|X_\tau - X_u|^\gamma}{|\tau - u|^{m+2}} d\tau du \right) = \\
&= 8 \cdot 4^{1/\gamma} \frac{m+2}{\gamma m} t^{m/\gamma} \left(\int_0^t \int_0^t \frac{|X_\tau - X_u|^\gamma}{|\tau - u|^{m+2}} d\tau du \right)^{(1-\gamma)/\gamma} \int_0^t \int_0^t \gamma \frac{|X_\tau - X_u|^{\gamma-1}}{|\tau - u|^{m+2}} \times \\
&\quad \times \left(\mathbb{1}_{[0, \tau]}(s) - \mathbb{1}_{[0, u]}(s) \right) d\tau du.
\end{aligned}$$

At this point, once the function Ψ is chosen, Monte Carlo simulation can be performed to estimate delta and gamma.

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